Retroactive Failure Correction for Strapdown Redundant Inertial Instruments

James C. Wilcox*
TRW Systems Group, Redondo Beach, Calif.

An algorithm is derived that eliminates the errors that build up in a redundant strapdown inertial navigator between the time that a soft failure occurs and the time that it is isolated and the failed instrument taken off line. (Hard failures are isolated before any erroneous data are used, by definition.) Although the algorithm is presented for the dodecahedron configuration as an example, it is applicable to any redundant system which uses least squares to combine the outputs of the unfailed instruments. Since only soft failures need be retroactively corrected, the errors involved are small and linearization can be used to reduce the data storage and computational requirements.

Introduction

REDUNDANCY management for the purpose of enhancing reliability is performed in three stages: failure detection in which the presence of a failure is discovered, failure isolation in which the failed instrument is identified, and failure correction in which a suitable reorganization of the system is accomplished. Many different failure detection, isolation, and correction (FDIC) algorithms for inertial navigators have appeared in the literature. Eight FDIC algorithms for the dodecahedron configuration[1,2] of six strapdown gyros and 6 strapdown accelerometers have been compared in a previous study.[3] We shall limit our discussion to the strapdown dodecahedron inertial navigation case although the methods to be developed are applicable to other cases in which the outputs of redundant devices are combined by least squares.

All 8 of the FDIC algorithms previously mentioned require a finite period of time to isolate a failure. If the failure causes a large enough error in the instrument output, it can be isolated in less than a minor cycle time so that no erroneous output is used and will be referred to as a hard failure. A failure that causes a smaller error in the instrument output cannot be isolated in less than a minor cycle time so that erroneous output is used. It will be referred to as a soft failure. Thus soft failures degrade system performance while hard failures do not.

An algorithm is desired that retroactively removes the effects of soft failures from the navigational solution at the time that the failure is finally isolated. Ideally, only the errors contributed by the instrument since the time of failure would be removed. However the time of failure is unknown, so all of the errors contributed by the failed instrument since initialization of the navigate mode will be removed. Since the FDIC cannot distinguish hard from soft failures, all of the error of the failed instrument is removed even if, in the case of hard failures, no error was introduced by the failure itself. Through the action of the least squares estimator, elimination of the error of one instrument magnifies the effects of the errors of the remaining unfailed instruments to the level they would have reached if the failed instrument had never been present. However this increase in error will usually be less than the increase caused by a soft failure prior to its isolation, because of the relatively high levels at which the FDIC thresholds must be set to avoid excessive false alarm rates.

For 6 instruments there are 1 case of 0 instruments failed, 6 cases of 1 instrument failed, and 15 cases of 2 instruments failed. For more than 2 failures isolation is not possible without external information. Thus there are a total of 22 cases for the gyros and 22 cases for the accelerometers for a total of 484 different cases for the strapdown package as a whole. It would be possible to carry along 484 different solutions corresponding to these 484 cases and switch to the appropriate one whenever a failure was detected. However such a "brute force" technique would consume much computer capacity. Each solution consists of an attitude quaternion, a position vector, and a velocity vector requiring 10 component differential equations per solution for a total of 4840 component differential equations altogether.

Since it is only the soft failures which are a problem, the errors are small, and a linearized solution is possible. We shall derive such a solution that will provide retroactive failure correction regardless of the trajectory of the vehicle and regardless of the order in which the instruments fail. As we shall see, only 135 component differential equations are required, a reduction of 97.2%. Future aerospace computers should be able to perform such calculations.

Derivation

To keep the notation from becoming too complex, we shall adopt the summation convention: 1) Each literal index which occurs once in a product assumes all of its possible values; 2) Each literal index which occurs twice in a product is a summation index, where the summation is to be carried out over all possible values; 3) Latin indices run from 1–3 and Greek indices run from 1–6. Thus

$$y_\mu = A_{\mu j} x_j \tag{1}$$

and

$$y_\mu = \sum_{j=1}^{3} A_{\mu j} x_j \, , \quad \mu = 1,2,\ldots 6 \tag{2}$$

mean the same thing: a 3-vector multiplied by a 6 by 3 matrix to give a 6-vector. Superscripts are used to indicate the coordinate system in which the components of a 3-vector are expressed: I inertial coordinate system, P strapdown package coordinate system. We wish to obtain linearized differential equations for the error states (position, velocity, and orientation) in terms of the computed trajectory and the error sources.

Presented as Paper 74-893 at the AIAA Mechanics and Control of Flight Conference, Anaheim, Calif., August 5–9, 1974; submitted August 22, 1974; revision received January 20, 1975.

Index categories: Reliability, Quality Control, and Maintainability; Navigation, Control, and Guidance Theory.

*Senior Staff Engineer, Navigation and Computer Systems Laboratory. Member AIAA.

The accelerometer and gyro outputs are given by

$$a_\mu = A_{\mu j}\alpha^P{}_j + \lambda_\mu \tag{3}$$

$$g_\mu = G_{\mu j}\omega^P{}_j + \eta_\mu \tag{4}$$

where a_μ = 6-vector of accelerometer outputs, g_μ = 6-vector of gyro outputs, $A_{\mu j}$ = 6 by 3 matrix of accelerometer input axis direction cosines with respect to package axes, $G_{\mu j}$ = 6 by 3 matrix of gyro input axis direction cosines with respect to package axes, $\alpha^P{}_j$ = nongravitational acceleration 3-vector, $\omega^P{}_j$ = angular velocity 3-vector, λ_μ = 6-vector of accelerometer errors, η_μ = 6-vector of gyro errors.

Typical nominal values for A and G are

$$A = G = \begin{bmatrix} c & s & 0 \\ c & -s & 0 \\ 0 & c & s \\ 0 & c & -s \\ s & 0 & c \\ -s & 0 & c \end{bmatrix} \tag{5}$$

where the rows are unit vectors perpendicular to the faces of the dodecahedron, and where

$$c = [0.5 + (0.05)^{1/2}]^{1/2} = \cos\alpha \tag{6}$$

$$s = [0.5 - (0.05)^{1/2}]^{1/2} = \sin\alpha \tag{7}$$

$$\tan 2\alpha = 2 \tag{8}$$

and 2α is the angle between any pair of instrument input axes in the dodecahedron configuration. In practice, A and G will differ somewhat from Eq. (5) and from each other because of manufacturing tolerances.

Some method of failure detection and isolation decides which instruments, if any, are failed. Let $\bar A$ and $\bar G$ be matrices which are equal to A and G except that the rows corresponding to failed instruments are set equal to zero. Failure correction (not retroactive as yet) is accomplished by the least-squares solution of Eqs. (3) and (4) for α and ω with $\bar A$ and $\bar G$ replacing A and G.

$$\hat\alpha^P{}_i = \bar B_{i\mu} a_\mu \tag{9}$$

$$\hat\omega^P{}_i = \bar H_{i\mu} g_\mu \tag{10}$$

where $\bar B$ and $\bar H$ are the least squares solution matrices

$$\bar B_{i\mu} = \bar P_{ij}\bar A_{\mu j} \tag{11}$$

$$\bar H_{i\mu} = \bar Q_{ij}\bar G_{\mu j} \tag{12}$$

where $\bar P$ and $\bar Q$ are the least squares covariance matrices

$$\bar P_{ij}\bar A_{\mu j}\bar A_{\mu k} = \delta_{ik} \tag{13}$$

$$\bar Q_{ij}\bar G_{\mu j}\bar G_{\mu k} = \delta_{ik} \tag{14}$$

and where δ_{ik} is the Kronecker delta. The computed angular velocity $\hat\omega$ is used to compute the package orientation direction cosine matrix by the strapdown equation

$$\dot{\hat M}_{ij} = -\epsilon_{ik\ell}\hat\omega^P{}_k \hat M_{\ell j} \tag{15}$$

where M is an orthogonal matrix that transforms a 3-vector from inertial to package axes.

$$x^P{}_i = M_{ij} x^I{}_j \tag{16}$$

and ϵ_{ijk} is the Levi-Civita epsilon. The actual direction cosine matrix is given by

$$\dot M_{ij} = -\epsilon_{ik\ell}\omega^P{}_k M_{\ell j} \tag{17}$$

The computed nongravitational acceleration $\hat\alpha$ is transformed into inertial coordinates using $\hat M$, and added to the computed gravitational acceleration to get the total computed acceleration

$$\dot{\hat v}^I{}_i = \hat\gamma^I{}_i + \hat M_{ji}\hat\alpha^P{}_j \tag{18}$$

where $\hat v^I{}_i$ = computed velocity 3-vector, $\hat\gamma^I{}_i$ = computed gravitational acceleration 3-vector. The actual total acceleration is

$$\dot v^I{}_i = \gamma^I{}_i + M_{ji}\alpha^P{}_j \tag{19}$$

The computed and actual velocity vectors are

$$\dot{\hat r}^I{}_i = \hat v^I{}_i \tag{20}$$

$$\dot r^I{}_i = v^I{}_i \tag{21}$$

Let us define the position, velocity, angular velocity, gravitational acceleration, and nongravitational acceleration errors as

$$\tilde r_i = \hat r_i - r_i \tag{22}$$

$$\tilde v_i = \hat v_i - v_i \tag{23}$$

$$\tilde\omega_i = \hat\omega_i - \omega_i \tag{24}$$

$$\tilde\gamma_i = \hat\gamma_i - \gamma_i \tag{25}$$

$$\tilde\alpha_i = \hat\alpha - \alpha_i \tag{26}$$

Orientation error is somewhat more complicated. We define the misorientation matrix as

$$\tilde M_{ik} = M_{ji}\hat M_{jk} \tag{27}$$

To first order, the misorientation matrix can be expressed in terms of the small angle misorientation vector ψ^I.

$$\tilde M_{ik} = \delta_{ik} - \epsilon_{ijk}\psi^I{}_j \tag{28}$$

From Eqs. (27) and (28)

$$M_{\ell i} = \hat M_{\ell k}(\delta_{ik} - \epsilon_{ijk}\psi^I{}_j) \tag{29}$$

From Eqs. (18–29)

$$\dot{\tilde r}^I{}_i = \tilde v^I{}_i \tag{30}$$

$$\dot{\tilde v}^I{}_i = \tilde\gamma^I{}_i + \hat M_{ji}\hat\alpha^P{}_j - \hat M_{\ell k}(\delta_{ik} - \epsilon_{ijk}\psi^I{}_j)(\hat\alpha^P{}_\ell - \tilde\alpha^P{}_\ell) \tag{31}$$

If we consider all error terms as first order and discard the second-order term Eq. (31) becomes

$$\dot{\tilde v}^I{}_i = \tilde\gamma^I{}_i + \hat M_{ji}\tilde\alpha^P{}_j + \epsilon_{ijk}\psi^I{}_j\hat M_{\ell k}\hat\alpha^P{}_\ell \tag{32}$$

We shall now treat in turn each term on the right hand side of Eq. (32). For the first term let us make the first-order approximation

$$\tilde\gamma^I{}_i = \hat\gamma^I{}_{i,j}\tilde r^I{}_j \tag{33}$$

where the comma indicates partial differentiation with respect to position.

$$a_{,k} = \partial a / \partial r_k \tag{34}$$

If we have an inverse square central force field

$$\hat{\gamma}^I{}_i = -\mu \hat{r}^I{}_i \,/\, \hat{r}^3 \tag{35}$$

where μ is the gravitational constant and

$$\hat{r}^2 = \hat{r}^I{}_i \hat{r}^I{}_i \tag{36}$$

then

$$\hat{\gamma}^I{}_{i,j} = -\frac{\mu}{\hat{r}^3}\left(\delta_{ij} - 3\,\frac{\hat{r}^I{}_i \hat{r}^I{}_j}{\hat{r}^2}\right) \tag{37}$$

A more complex expression can be used when Eq. (35) is inadequate, as for a translunar mission. For the second term, we see from Eqs. (26) and (9) that

$$\tilde{\alpha}^P{}_i = \bar{B}_{i\mu} a_\mu - \alpha^P{}_i \tag{38}$$

For the third term, we note that by Eq. (16),

$$\hat{\alpha}^I{}_i = \hat{M}_{ji} \hat{\alpha}^P{}_j \tag{39}$$

Thus we may write Eq. (32) as

$$\tilde{\dot{v}}^I{}_i = \hat{\gamma}^I{}_{i,j} \tilde{r}^I{}_j + \hat{M}_{ji}(\bar{B}_{j\mu} a_\mu - \alpha^P{}_j) + \epsilon_{ijk} \psi^I{}_j \hat{\alpha}^I{}_k \tag{40}$$

In Eqs. (30) and (40) we have differential equations for the position and velocity error states respectively. We now require a differential equation for the orientation error states. It can be shown that, if the small angle misorientation vector is expressed in inertial components, its differential equation is

$$\dot{\psi}^I{}_i = \tilde{\omega}^I{}_i \tag{41}$$

Applying Eq. (16) to Eq. (41) gives

$$\dot{\psi}^I{}_i = \hat{M}_{ji} \tilde{\omega}^P{}_j \tag{42}$$

From Eqs. (10) and (24) we see that

$$\tilde{\omega}^P{}_j = \bar{H}_{j\mu} g_\mu - \omega^P{}_j \tag{43}$$

so that Eq. (42) becomes

$$\dot{\psi}^I{}_i = \hat{M}_{ji}(\bar{H}_{j\mu} g_\mu - \omega^P{}_j) \tag{44}$$

We now have three differential equations, Eqs. (30), (40), and (44) which together give the position, velocity, and orientation errors in terms of the gyro and accelerometer outputs, the actual acceleration and angular velocity, and the failure correction matrices $\bar{B}$ and $\bar{H}$. We cannot solve these equations in the onboard computer because we do not know the actual acceleration α^P and angular velocity ω^P. However the equations are linear so that we can obtain solutions for the difference between the errors for different values of $\bar{B}$ or $\bar{H}$. These solutions do not involve α^P or ω^P, since they cancel out. Indicating the old solution by 1 and the new solution by 2 gives

$$\dot{\tilde{r}}^I{}_i(2) - \dot{\tilde{r}}^I{}_i(1) = \tilde{v}^I{}_i(2) - \tilde{v}^I{}_i(1) \tag{45}$$

$$\begin{aligned}\dot{\tilde{v}}^I{}_i(2) - \dot{\tilde{v}}^I{}_i(1) &= \hat{\gamma}^I{}_{i,j}[\tilde{r}^I{}_j(2) - \tilde{r}^I{}_j(1)] \\ &+ \hat{M}_{ji}[\bar{B}_{j\mu}(2) - \bar{B}_{j\mu}(1)]a_\mu + \epsilon_{ijk}[\psi^I{}_j(2) - \psi^I{}_j(1)]\hat{\alpha}^I{}_k\end{aligned} \tag{46}$$

$$\dot{\psi}^I{}_i(2) - \dot{\psi}^I(1) = \hat{M}_{ji}[\bar{H}_{j\mu}(2) - \bar{H}_{j\mu}(1)]g_\mu \tag{47}$$

Now we let

$$\Delta r^I{}_i = \tilde{r}^I{}_i(2) - \tilde{r}^I{}_i(1) \tag{48}$$

$$\Delta v^I{}_i = \tilde{v}^I{}_i(2) - \tilde{v}^I{}_i(1) \tag{49}$$

$$\Delta\psi^I{}_i = \psi^I{}_i(2) - \psi^I{}_i(1) \tag{50}$$

From Eqs. (45 – 50)

$$\Delta \dot{r}^I{}_i = \Delta v^I{}_i \tag{51}$$

$$\begin{aligned}\Delta \dot{v}^I{}_i &= \hat{\gamma}^I{}_{i,j} \Delta r^I{}_j + \hat{M}_{ji}[\bar{B}_{j\mu}(2) - \bar{B}_{j\mu}(1)]a_\mu \\ &+ \epsilon_{ijk} \Delta\psi^I{}_j \hat{\alpha}^I{}_k\end{aligned} \tag{52}$$

$$\Delta\dot{\psi}^I{}_i = \hat{M}_{ji}[\bar{H}_{j\mu}(2) - \bar{H}_{j\mu}(1)]g_\mu \tag{53}$$

In Eqs. (51 – 53) we have a set of time varying linear differential equations in which Δr^I, Δv^I, and $\Delta\psi^I$ are the dependent variables; $\hat{\gamma}^I$, $\hat{M}$, and $\hat{\alpha}^I$ are time varying coefficients; $\bar{B}(1)$, $\bar{B}(2)$, $\bar{H}(1)$, and $\bar{H}(2)$ are constants; and the 6-vectors a and g are the driving functions. The number of elements to be integrated in the eventual solution will be halved if the driving functions can be replaced by 3-vectors. We can accomplish this reduction by introducing the inconsistency states.[3]

If there are n instruments of a given type, it is possible to define $n-3$ signals that are linear combinations of the outputs of the n instruments, are independent of the inputs to the instruments, and depend only on the instrument errors. These signals are used by the FDIC. It is possible for the instrument errors to be such that the resultant outputs are the same as the outputs that would be produced by an actual motion of the instrument package so that the $n-3$ signals remain zero. It is only when the instrument outputs are inconsistent with an actual motion of the package that the signals are nonzero, hence their name. One example of the infinite number of possible sets of inconsistency states will be derived, starting with the residuals of the least-squares solution with no gyros failed, since we expect to find the information needed to transition from one solution to another in the residuals.

Consider the failure correction equation for zero failures for the accelerometers. From Eqs. (9) and (11)

$$\hat{\alpha}^P{}_i = P_{ij} A_{\mu j} a_\mu \tag{54}$$

The residuals of this least squares solution are the differences between the actual accelerometer outputs and the accelerometer outputs calculated from the estimate $\hat{\alpha}^P$.

$$d_\nu = a_\nu - A_{\nu i} \hat{\alpha}^P{}_i \tag{55}$$

$$d_\nu = (\delta_{\nu\mu} - A_{\nu i} P_{ij} A_{\mu j}) a_\mu \tag{56}$$

$$d_\nu = D_{\nu\mu} a_\mu \tag{57}$$

From Eqs. (13, 56, and 57) we see that

$$D_{\nu\mu} A_{\mu i} = A_{\nu i} D_{\nu\mu} = 0 \tag{58}$$

Thus D is a symmetric matrix of no more than rank 3. It is this fact that will let us replace the 6-vectors with 3-vectors. Suppose D is diagonalized by a 6 by 6 orthogonal matrix Φ

$$D'_{\alpha\beta} = \Phi_{\alpha\nu} \Phi_{\beta\mu} D_{\nu\mu} \tag{59}$$

then

$$D_{\nu\mu} = \Phi_{\alpha\nu} \Phi_{\beta\mu} D'_{\alpha\beta} \tag{60}$$

By taking the square roots of the diagonal elements of D' we get

$$D'_{\gamma\alpha}{}^{1/2} D'_{\gamma\beta}{}^{1/2} = D'_{\alpha\beta} \tag{61}$$

From Eqs. (60) and (61)

$$D_{\nu\mu} = (\Phi_{\alpha\nu} D'_{\gamma\alpha}{}^{1/2})(\Phi_{\beta\mu} D'_{\gamma\beta}{}^{1/2}) \tag{62}$$

However three of the diagonal elements of D' are zero since D is of rank 3, so we may write

$$D_{\nu\mu} = (\Phi_{i\nu} D'_{ki}{}^{1/2})(\Phi_{j\mu} D'_{kj}{}^{1/2}) \tag{63}$$

Let the 3 by 6 matrix C be

$$C_{k\nu} = \Phi_{i\nu} D'_{ki}{}^{1/2} \tag{64}$$

Then

$$D_{\nu\mu} = C_{k\nu} C_{k\mu} \tag{65}$$

and from Eqs. (56) and (65)

$$C_{k\nu} C_{k\mu} = \delta_{\nu\mu} - A_{\nu i} P_{ij} A_{\mu j} \tag{66}$$

Similarly for the gyros

$$K_{k\rho} K_{k\mu} = \delta_{\nu\mu} - G_{\nu i} Q_{ij} G_{\mu j} \tag{67}$$

For nominal A and G we may choose C and K as

$$C = K = 2^{-1/2} \begin{bmatrix} s & s & 0 & 0 & -c & c \\ -c & c & s & s & 0 & 0 \\ 0 & 0 & -c & c & s & s \end{bmatrix} \tag{68}$$

The inconsistency states are given by

$$y_i = C_{i\mu} a_\mu \tag{69}$$

for the accelerometers and by

$$z_i = K_{i\mu} g_\mu \tag{70}$$

for the gyros. We want to use Eq. (69) and (70) to replace a_μ and g_μ in Eqs. (52) and (53). Let us write down the answer and show that it is correct. But first we must recognize that

$$\bar{A}_{\mu i} A_{\mu j} = \bar{A}_{\mu i} \bar{A}_{\mu j} \tag{71}$$

since the nonzero elements of A which are zero in $\bar{A}$ are multiplied by the zeros in $\bar{A}$. The middle bracket in Eq. (52) becomes

$$\begin{gathered}
[\bar{B}_{j\mu}(2) - \bar{B}_{j\mu}(1)] C_{k\mu} y_k = \\
[\bar{B}_{j\mu}(2) - \bar{B}_{j\mu}(1)] C_{k\mu} C_{k\nu} a_\nu = \\
[\bar{B}_{j\mu}(2) - \bar{B}_{j\mu}(1)][\delta_{\mu\nu} - A_{\mu i} P_{ik} A_{\nu k}] a_\nu = \\
[\bar{B}_{j\mu}(2) - \bar{B}_{j\mu}(1)] a_\mu - \\
[\bar{P}_{jk}(2) \bar{A}_{\mu k}(2) - \bar{P}_{jk}(1) \bar{A}_{\mu k}(1)] A_{\mu i} P_{ik} A_{\nu k} a_\nu = \\
[\bar{B}_{j\mu}(2) - \bar{B}_{j\mu}(1)] a_\mu - [\delta_{ji} - \delta_{ji}] P_{ik} A_{\nu k} a_\nu = \\
[\bar{B}_{j\mu}(2) - \bar{B}_{j\mu}(1)] a_\mu
\end{gathered} \tag{72}$$

Similarly, the bracket in Eq. (53) is

$$[\bar{H}_{j\mu}(2) - \bar{H}_{j\mu}(1)] K_{k\mu} z_k = [\bar{H}_{j\mu}(2) - \bar{H}_{j\mu}(1)] g_\mu \tag{73}$$

and we have succeeded in replacing the 6-vectors by 3-vectors. Applying Eqs. (72) and (73) to Eqs. (51 – 53) gives

$$\Delta \dot{r}^I_i = \Delta v^I_i \tag{74}$$

$$\Delta \dot{v}^I_i = \hat{\gamma}^I_{i,j} \Delta r^I_j + \hat{M}_{ji} [\bar{B}_{j\mu}(2) - \bar{B}_{j\mu}(1)] C_{k\mu} y_k + \epsilon_{ijk} \Delta\psi^I_j \hat{\alpha}^I_k \tag{75}$$

$$\Delta \dot{\psi}^I_i = \hat{M}_{ji} [\bar{H}_{j\mu}(2) - \bar{H}_{j\mu}(1)] K_{k\mu} z_k \tag{76}$$

These are the error equations that have to be solved. $\hat{\gamma}$, $\hat{M}$, and $\hat{\alpha}$ are available from the strapdown navigation algorithm while $\bar{H}$, $\bar{B}$, C, K, y, and z are available from the FDIC algorithm. $\bar{H}$ or $\bar{B}$, however, each can take on 22 different forms depending upon the failures that occur. Thus we define the 3 by 3 by 3 arrays R, S, T, U, and V.

$$\begin{aligned} \Delta r^I_i = {} & R_{jik} [\bar{H}_{j\mu}(2) - \bar{H}_{j\mu}(1)] K_{k\mu} \\ & + U_{jik} [\bar{B}_{j\mu}(2) - \bar{B}_{j\mu}(1)] C_{k\mu} \end{aligned} \tag{77}$$

$$\begin{aligned} \Delta v^I_i = {} & S_{jik} [\bar{H}_{j\mu}(2) - \bar{H}_{j\mu}(1)] K_{k\mu} \\ & + V_{jik} [\bar{B}_{j\mu}(2) - \bar{B}_{j\mu}(1)] C_{k\mu} \end{aligned} \tag{78}$$

$$\Delta \psi^I_i = T_{jik} [\bar{H}_{j\mu}(2) - H_{j\mu}(1)] K_{k\mu} \tag{79}$$

Then from Eqs. (74-79) we see that

$$\dot{R}_{jik} = S_{jik} \tag{80}$$

$$\dot{S}_{jik} = \hat{\gamma}^I_{i,\ell} R_{j\ell k} + \epsilon_{im\ell} T_{jmk} \hat{\alpha}^I_\ell \tag{81}$$

$$\dot{T}_{jik} = \hat{M}_{ji} z_k \tag{82}$$

$$\dot{U}_{jik} = V_{jik} \tag{83}$$

$$\dot{V}_{jik} = \hat{\gamma}^I_{i,\ell} U_{j\ell k} + \hat{M}_{ji} y_k \tag{84}$$

The 135 differential equations (80 – 84) are integrated throughout the mission. At any time they provide the information necessary to calculate the change in the errors in the states caused by a change in $\bar{B}$ or $\bar{H}$ using Eqs. (77 – 79). The old and new values of $\bar{B}$ or $\bar{H}$ are readily available from the FDIC algorithm.

To see how the change of $\bar{B}$ or $\bar{H}$ affects the computed inertial position we write the identity

$$\hat{r}^I_i(2) = \hat{r}^I_i(1) + [\hat{r}^I_i(2) - \hat{r}^I_i(1)] \tag{85}$$

Since the actual solution does not change with change of $\bar{B}$ or $\bar{H}$ we have, from Eq. (22)

$$\hat{r}^I(2) = \hat{r}^I_i(1) + [\tilde{r}^I_i(2) - \tilde{r}^I_i(1)] \tag{86}$$

By Eq. (48)

$$\hat{r}^I_i(2) = \hat{r}^I_i + \Delta r^I_i \tag{87}$$

Similarly the inertial velocity reset is

$$\hat{v}^I_i(2) = \hat{v}^I_i(1) + \Delta v^I_i \tag{88}$$

For orientation, we write, from Eq. (27)

$$\hat{M}_{ij}(2) = M_{ik} \tilde{M}_{kj}(2) \tag{89}$$

$$\hat{M}_{ij}(1) = M_{ik} \tilde{M}_{kj}(1) \tag{90}$$

Combining Eqs. (89) and (90) gives

$$\hat{M}_{ij}(2) = \hat{M}_{ik}(1) \tilde{M}_{\ell k}(1) \tilde{M}_{\ell j}(2) \tag{91}$$

From Eqs. (29) and (91)

$$\hat{M}_{ij}(2)=\hat{M}_{ik}(1)[\delta_{\ell k}-\epsilon_{\ell mk}\psi^{I}{}_{m}(1)]$$
$$[\delta_{\ell j}-\epsilon_{\ell mj}\psi^{I}{}_{m}(2)] \tag{92}$$

Discarding the second-order term,

$$\hat{M}_{ij}(2)=\hat{M}_{ik}(1)[\delta_{kj}-\epsilon_{kmj}\psi^{I}{}_{m}(2)-\epsilon_{jmk}\psi^{I}{}_{m}(1)] \tag{93}$$

From Eq. (50)

$$\hat{M}_{ij}(2)=\hat{M}_{ik}(1)[\delta_{kj}-\epsilon_{kmj}\Delta\psi^{I}{}_{m}] \tag{94}$$

Equation (94) is the first-order solution to the reset. However, the actual $\Delta\psi^{I}$ is not infinitesimal so that the matrix in brackets is not actually orthogonal. To obtain an expression for an orthogonal reset matrix, we assume that a finite rotation equal to the magnitude of $\Delta\psi^{I}$ is made about an axis parallel to $\Delta\psi^{I}$. Let the magnitude of $\Delta\psi^{I}$ be

$$\Delta\psi=(\Delta\psi^{I}{}_{i}\Delta\psi^{I}{}_{i})^{1/2} \tag{95}$$

Then the reset equation is

$$\hat{M}_{ij}(2)=\hat{M}_{ik}(1)[\delta_{kj}-\frac{\sin\Delta\psi}{\Delta\psi}\epsilon_{kmj}\Delta\psi^{I}{}_{m}$$
$$+\frac{1-\cos\Delta\psi}{\Delta\psi^{2}}(\Delta\psi^{I}{}_{k}\Delta\psi^{I}{}_{j}-\Delta\psi^{2}\delta_{kj})] \tag{96}$$

An analogous solution is easily found if the orientation is computed in terms of quaternions instead of direction cosines. When an accelerometer failure occurs, only the terms involving $\bar{B}$ in Eqs. (77–79) are nonzero. Let us define the 3 by 3 temporary storage matrix W as

$$W_{jk}=[\bar{B}_{j\mu}(2)-\bar{B}_{j\mu}(1)]C_{k\mu} \tag{97}$$

Then from Eqs. (77, 78, 87, 88, and 97)

$$\hat{r}^{I}{}_{i}(2)=\hat{r}^{I}{}_{i}(1)+U_{jik}W_{jk} \tag{98}$$

$$\hat{v}^{I}(2)=\hat{v}^{I}{}_{i}(1)+V_{jik}W_{jk} \tag{99}$$

When a gyro failure occurs, only the terms involving $\bar{H}$ in Eqs. (77–79) are nonzero. Let

$$W_{jk}=[\bar{H}_{j\mu}(2)-\bar{H}_{j\mu}(1)]K_{k\mu} \tag{100}$$

Then from Eqs. (77 79, 87, 88, 95, 96, and 100)

$$\hat{r}^{I}{}_{i}(2)=\hat{r}^{I}(1)+R_{jik}W_{jk} \tag{101}$$

$$\hat{v}^{I}{}_{i}(2)=\hat{v}^{I}{}_{i}(1)+S_{jik}W_{jk} \tag{102}$$

$$\Delta\psi^{I}{}_{i}=T_{jik}W_{jk} \tag{103}$$

$$\Delta\psi=(\Delta\psi^{I}{}_{i}\Delta\psi^{I}{}_{i})^{1/2} \tag{104}$$

$$\bar{M}_{ij}(2)=\bar{M}_{ik}(1)[\delta_{kj}-\frac{\sin\Delta\psi}{\Delta\psi}\epsilon_{kmj}\Delta\psi^{I}_{m}$$
$$+\frac{1-\cos\Delta\psi}{\Delta\psi^{2}}(\Delta\psi^{I}_{k}\Delta\psi^{I}_{j}-\Delta\psi^{2}\delta_{kj})] \tag{105}$$

Summary

Equations (80–84) are integrated throughout the mission. Equations (97–99) are performed whenever an accelerometer failure is isolated. Equations (100–105) are performed whenever a gyro failure is isolated.

References

[1] Ephgrave, J.T., "Optimum Redundant Configurations of Inertial Sensors," TOR-1001 (9990)-5, Sept. 1966, Aerospace Corporation, El Segundo, Calif.

[2] Gilmore, J.P., "A Non-Orthogonal Gyro Configuration," M. Sc. thesis, Jan. 1967, Dept. of Aeronautics and Astronautics MIT, Cambridge, Mass.

[3] Wilcox, J.C., "Competitive Evaluation of Failure Detection Algorithms for Strapdown Redundant Inertial Instruments," *Journal of Spacecraft and Rockets*, Vol. 11, July 1974, pp. 525–530.